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On "P" property and the column-W property

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ABSTRACT

P-matrices play an important role in the well-posedness of a linear complementarity problem (LCP). Similarly, the well-posedness of a horizontal linear complementarity problem (HLCP) is closely related to the column-W property of a matrix k-tuple.

In this paper we first consider the problem of generating P-matrices from a given pair of matrices. Given a matrix pair (D, F) where D is a square matrix of order m and matrix F has m rows, "what are the conditions under which there exists a matrix G such that $(D + FG)$ is a P-matrix?". We obtain necessary and sufficient conditions for the special case when the column rank of F is $m - 1$. A decision algorithm of complexity $O(m^2)$ to check whether the given pair of matrices (D, F) is P-matrisable is obtained. We also obtain a necessary and an independent sufficient condition for the general case when rank(F) is less than $m - 1$.

We then generalise the P-matrix generating problem to the generation of matrix k-tuples satisfying the column-W property from a given matrix $(k + 1)$-tuple. That is, given a matrix $(k + 1)$-tuple $(D_1, \ldots, D_k, F)$, where D_js are square matrices of order m and F is a matrix having m rows, we determine the conditions under which the matrix k-tuple $(D_1 + FG_1, \ldots, D_k + FG_k)$ satisfies the column-W property. As in the case of P-matrices we obtain necessary and sufficient conditions for the case when rank$(F) = m - 1$. Using these conditions a decision algorithm of complexity $O(km^2)$ to check whether the given matrix $(k + 1)$-tuple is column-W matrisable is obtained. Then for the case when rank(F) is less than $m - 1$, we obtain a necessary and an independent sufficient condition.

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For a special sub-class of P-matrices we give a polynomial time decision algorithm for P-matrisability. Finally, we obtain a geometric characterisation of column-W property by generalising the well known separation theorem for P-matrices.

1. Introduction

The class of P-matrices arise in a wide range of applications [1,2]. Among the many areas of research related to P-matrices are the generation, detection of P-matrices and transformations preserving the P-property [3]. P-matrices also arise quite frequently in systems theory. These include Hermitian positive definite matrices, the M-matrices, totally positive matrices and real diagonally dominant matrices with positive diagonal entries. For instance, electrical networks that have resistances with positive values give rise to diagonally dominant matrices with positive diagonal entries.

One characterisation of P-matrices is obtained by using the principal minors. Since this characterisation is commonly, we treat this as a definition for P-matrices in this paper. The P-matrix class is defined as follows:

Definition 1.1 [1]. A square matrix is said to be from the P-matrix class if all its principal minors are positive.

For other characterisations of P-matrices see [1,4]. We denote by $\mathbb{P}$ the set of all P-matrices of a particular order (which is understood from the context). We now introduce notation used in this paper, followed by motivation for looking at the problem of P-matrisability.

1.1. Notation

We denote the set of natural numbers with $\mathbb{N}$, real numbers with $\mathbb{R}$. $\overline{m}$ stands for the finite set $\{1, 2, \ldots, m\}$. The number of elements in a finite set α is denoted by $|\alpha|$. $\operatorname{sgn}(\cdot)$ denotes the sign function which takes $+1$ for positive and -1 for negative arguments. Further we assume $\operatorname{sgn}(0) = 0$. We define the absolute value of any quantity a as $\operatorname{abs}(a) = \operatorname{sgn}(a)a$.

Vectors are denoted by bold small case letters. The ith component of a vector $\mathbf{v} \in \mathbb{R}^m$ is denoted by $\mathbf{v}_i$. $\mathbb{R}^{m\times p}$ represents the set of all real matrices with m rows and p columns. I denotes the identity matrix (whose dimensions would be clear from the context). The notation $[0, I]$ denotes the set of all diagonal matrices whose diagonal entries are in the interval $[0, 1]$.

Let $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^m\}$ denote the standard (orthonormal) basis of $\mathbb{R}^m$. Given $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$ their dot product is given by $\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{i=1}^m \mathbf{v}_i\mathbf{w}_i$.

Let $\alpha \subseteq \overline{m}$, $\beta \subseteq \overline{p}$ and $M \in \mathbb{R}^{m\times p}$. $M[\alpha, \beta]$ then denotes the sub-matrix of M consisting of rows indexed by α and columns indexed by β. $M[\bullet, \beta]$ is the sub-matrix of M containing all the rows of M and all columns indexed by β, whereas $M[\alpha, \bullet]$ is the sub-matrix of M containing rows indexed by α and all columns of M. The ith row of M is given by $M[i, \bullet]$ and the jth column of M by $M[\bullet, j]$. $M[\alpha, \alpha]$ denotes a principal sub-matrix. The determinant of a principal sub-matrix is called a principal minor of that matrix.

1.2. Motivation for P-matrisability problem

The study of P-matrices also arose from the investigation of well-posedness of linear complementarity problem (LCP). An LCP$(\mathbf{q}, M)$ is defined as follows:

Problem 1.1 [1]. Given a vector $\mathbf{q} \in \mathbb{R}^m$ and a square matrix $M \in \mathbb{R}^{m\times m}$, find $\mathbf{z} \in \mathbb{R}^m$ such that $\mathbf{z} \succeq 0$, $\mathbf{q} + M\mathbf{z} \succeq 0$ and $\mathbf{z} \perp (\mathbf{q} + M\mathbf{z})$.

The $\perp$ symbol denotes orthogonality between two vectors. That is, $\sum_{i=1}^{m} \mathbf{z}_i(\mathbf{q}+M\mathbf{z})_i = 0$. The vector inequality $\succeq$ should be taken component-wise, that is, $\mathbf{z} \succeq 0$ means that $\mathbf{z}_i \geq 0$ for all $i = 1, 2, \ldots, m$. Therefore, a solution $\mathbf{z}$ of LCP($\mathbf{q}$, M) satisfies either [$\mathbf{z}_i \geq 0$ and $(\mathbf{q} + M\mathbf{z})_i = 0$] or [$\mathbf{z}_i = 0$ and $(\mathbf{q} + M\mathbf{z})_i \geq 0$].

An important question concerning LCP is the existence and uniqueness of solution $\mathbf{z}$ satisfying the constraints. In general, a system is said to be well-posed [5] if (a) a solution exists (b) the solution is unique and (c) the solution is continuously dependent on the data. For LCP we consider only the existence and uniqueness of solution. So formally, the well-posedness of an LCP is defined as follows:

Definition 1.2 [1]**.** An LCP($\mathbf{q}$, M) is said to be well-posed if for every $\mathbf{q}$ there exists a unique $\mathbf{z} \succeq 0$ satisfying $\mathbf{z} \succeq 0$, $\mathbf{q} + M\mathbf{z} \succeq 0$ and $\mathbf{z} \perp (\mathbf{q} + M\mathbf{z})$.

The following lemma relates P-matrices to well-posedness of LCP:

Lemma 1.1 [1]**.** *An LCP($\mathbf{q}$, M) is well-posed if and only if M is a P-matrix.*

The LCP has a wide range of applications. One such application is in piecewise linear systems (e.g., electrical networks with ideal diodes). Such networks are frequently represented using the framework of linear complementarity systems (LCS). Mathematically, an LCS is described by the following set of relations [6–9]:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + E\mathbf{w}(t), \tag{1a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) + F\mathbf{w}(t), \tag{1b}$$

$$0 \preceq \mathbf{u}(t) \perp \mathbf{y}(t) \succeq 0, \tag{1c}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$, $\mathbf{w}(t) \in \mathbb{R}^p$, and matrices A, B, C, D, E and F are of compatible dimensions. The vector $\mathbf{w}(t)$ represents the input to LCS.

Note that at every time instant, the constraints (1b) and (1c) above defines an LCP. Thus LCS can be thought of as a dynamic version of the (static) LCP defined in Problem 1.1 . Note that in (1b), the vector $C\mathbf{x}(t) + F\mathbf{w}(t)$ plays the role of $\mathbf{q}$, the matrix D represents M, $\mathbf{u}(t)$ is like $\mathbf{z}$ and $\mathbf{y}(t)$ is similar to $\mathbf{q} + M\mathbf{z}$ of some LCP($\mathbf{q}$, M).

An LCS is said to be well-posed whenever there exists a unique solution for all initial conditions and admissible inputs. It is clear from the mathematical relations of the LCS that its well-posedness is related to the existence and uniqueness of the solutions of its LCP constraints. In particular, the (dynamic) LCP arising from the LCS is well-posed if and only if D is a P-matrix. Hence if D is a P-matrix, the LCS (1) is well-posed.

Since, for a general LCS the matrix D may not be a P-matrix, the LCS (1) may not be well-posed. Hence one way to regularise this LCS is by applying feedback which modifies matrix D to a P-matrix. Application of port feedback $\mathbf{w}(t) = G\mathbf{u}(t) + \mathbf{v}(t)$ results in the LCS (1) being modified as follows:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + (B + EG)\mathbf{u}(t) + E\mathbf{v}(t), \tag{2a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + (D + FG)\mathbf{u}(t) + F\mathbf{v}(t), \tag{2b}$$

$$0 \preceq \mathbf{u}(t) \perp \mathbf{y}(t) \succeq 0. \tag{2c}$$

Thus if the matrix $(D + FG)$ is a P-matrix then LCP($C\mathbf{x}(t) + F\mathbf{w}(t)$, $D + FG$) is well-posed for all $\mathbf{x}(t)$, $\mathbf{w}(t)$ and hence the feedback LCS (2).

Motivated by the LCS problem, in this paper we first consider the problem of generating P-matrices from a pair of matrices. Let $D \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times p}$. The problem statement is: Given a matrix pair (D, F), determine the conditions under which there exists a matrix G such that $D + FG$ is a P-matrix. The pair (D, F) will be called P-matrisable if there exists G such that $D + FG$ is a P-matrix.

2. Properties of P-matrices

In this section, we gather together some useful properties of P-matrices from [1,3,10–12]. For ease of use in the subsequent sections, we state some of these results in the form of lemmas. The following properties of P-matrices are of crucial importance for this paper:

(i) Every principal sub-matrix of a P-matrix is again a P-matrix. This property follows from the definition.
(ii) Determinant of a P-matrix is non-zero and hence P-matrices are invertible.
(iii) The real eigenvalues of a P-matrix are all positive [1].
(iv) The set of P-matrices is invariant under permutation similarity transformations [3].
(v) Every P-matrix satisfies the strict separation property [4].
(vi) The set of all P-matrices forms a connected set in $\mathbb{R}^{m\times m}$. (Follows from the column-hull property defined later.)
(vii) The P-matrix class forms a non-convex and positive cone in the space of matrices under consideration, see Example 2.1 below.

The nonconvexity of the P-matrix class is seen from the following example:

Example 2.1. Let the matrices P_1, P_2 and P_3 be as defined below:

$$P_1 = \begin{bmatrix} 2 & 3.9 \\ 1 & 2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & 1 \\ 3.9 & 2 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 2 & 2.45 \\ 2.45 & 2 \end{bmatrix}.$$

It is clear that P_1 and P_2 are P-matrices whereas, $P_3 = 0.5P_1 + 0.5P_2$, which is a convex combination of P_1 ad P_2 is not a P-matrix.

P-matrices which are permutationally similar to upper block diagonal or lower block diagonal matrices are characterised by these diagonal blocks. The following lemma characterises P-matrices which are permutationally similar to upper and lower block diagonal matrices.

Lemma 2.1. *Consider a matrix $M \in \mathbb{R}^{m\times m}$. Let $\alpha \subset \overline{m}$ and $\beta = \overline{m} \setminus \alpha$. Suppose either $M[\alpha, \beta] = 0$ or $M[\beta, \alpha] = 0$. Then M is a P-matrix if and only if the principal sub-matrices $M[\alpha, \alpha]$ and $M[\beta, \beta]$ are P-matrices.*

Proof. This result follows from the definition of P-matrices. $\square$

The following lemma shows the relation between the P-matrices and matrix interval. Recall the definition of the matrix interval $[0, I]$ which denotes all diagonal matrices with diagonal entries in the interval $[0, 1]$.

Lemma 2.2 [13]**.** *Let $Q \in \mathbb{R}^{m\times m}$. Then $Q \in \mathbb{P}$ if and only if $Q\Lambda + (I - \Lambda)$ is invertible for all $\Lambda \in [0, I]$.*

The above property is also called the column hull property of a P-matrix. The following result which shows that the P-matrices form a connected set in $\mathbb{R}^{m^2}$ is crucial for the proof of the main theorem on P-matrisability.

Lemma 2.3. *Let $M \in \mathbb{R}^{m\times m}$. Let $\mathbf{e^i}$ be the ith standard basis vector. Let M_i be the matrix in which the ith column of M, that is, $M[\bullet, i]$ is replaced with $aM[\bullet, i] + b\mathbf{e^i}$ where $a, b \in \mathbb{R}$. Then M is a P-matrix if and only if M_i is a P-matrix for all $a, b \geq 0$ and with at least one of a, b non-zero.*

Proof. Suppose M_i is a P-matrix for all a, b satisfying the conditions given in the lemma and for all $i = 1, 2, \ldots, m$. Then for $b = 0$, $a = 1$ one has $M_i = M$ and so, M is a P-matrix.

Conversely, assume M is a P-matrix. Let R be the matrix obtained when the ith column $M[\bullet, i]$ of matrix M is replaced by $aM[\bullet, i]$ for $a > 0$. Now R is a P-matrix, because, multiplying a column of a matrix by a scalar will just scale the principal minors containing that column by the same factor. Also let S denote the matrix obtained after replacing the ith column $M[\bullet, i]$ of matrix M by $b\mathbf{e^i}$ for $b > 0$. From the properties of P-matrix it is clear that for $\alpha = \overline{m} \setminus \{i\}$ the sub-matrix $M[\alpha, \alpha]$ is a P-matrix. Therefore, by Lemma 2.1 , S is also a P-matrix as $b > 0$. Now consider any matrix M_i defined in the lemma. Note that multi-linearity of the determinant function ensures that any principal minor of M_i is the same as that of M (if $i \notin \alpha$), or is the sum of corresponding principal minors of R and S. That is, for $\alpha \subseteq \overline{m}$:

$$\det(M_i[\alpha, \alpha]) = \begin{cases} \det(M[\alpha, \alpha]) & \text{if } i \notin \alpha \\ a\det(M[\alpha, \alpha]) + b\det(S[\alpha, \alpha]) & \text{otherwise.} \end{cases}$$

Further, for $a = 0, b = 1$, one has $M_i = S$ and when $a = 1$ and $b = 0$, the matrix $M_i = R = M$. Therefore, M_i is a P-matrix for $a, b \geq 0$, with at least one of a, b non-zero. $\square$

3. P-matrisability

Let $D \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times p}$. The existence of a matrix G such that $D + FG$ is a P-matrix is explored in this section. Assume without loss of generality that F is full column rank. The following are some trivial/simple cases for existence and non-existence of matrix G for solving the P-matrisability problem:

1. If $p = m$ (i.e., number of inputs and complementarity variables are equal) and F is invertible, then for every matrix D there exists a matrix G such that $D + FG$ is a P-matrix. Note that in this case, the set of matrices $D + FG$ will include all square matrices of order m and hence every element in the class of P-matrices.
2. If $\text{rank}[D\,F] < m$ then there exists no matrix G such that $D + FG$ is a P-matrix, as P-matrices are invertible.
3. Spectral properties of P-matrices: It is known that the real eigenvalues of P-matrices are always positive [1]. Let the eigenvectors of non positive eigenvalues of D be called the “bad part” of D’s spectrum. Then there exist no G such that $D + FG$ is a P-matrix, if the “bad part” of D’s spectrum is not contained in the controllable space of the pair (D, F).
4. Suppose the pair (D, F) is controllable. Then one can choose G such that all the eigenvalues of $D + FG$ are positive. But an arbitrary matrix with positive eigenvalues need not be a P-matrix. For example the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

has both the eigenvalues at 1. But A is not a P-matrix as the principal minor $A[1, 1]$ is negative. Thus the problem of obtaining P-matrices is more involved than the pole placement problem encountered in control theory.

3.1. P-matrisability by hyperplanes

More involved cases of the P-matrisability problem are analysed now. Assume that $\text{rank}(F) = m-1$ (system with under-actuation degree one with respect to complementarity variables). Let $\mathscr{H}$ denote the hyperplane spanned by the columns of F. It divides the m dimensional space into two open half spaces. This hyperplane $\mathscr{H}$ is also defined by a normal vector $\mathbf{n}$ as follows: $\mathscr{H} = \{\mathbf{v} \in \mathbb{R}^m | \langle \mathbf{n}, \mathbf{v}\rangle = 0\}$. For vectors $\mathbf{v}$ not lying on the hyperplane $\mathscr{H}$, the inner product $\langle \mathbf{n}, \mathbf{v}\rangle \neq 0$ and therefore, the sign of $\langle \mathbf{n}, \mathbf{v}\rangle$ determines the half-space in which $\mathbf{v}$ lies.

Let $\mathbf{e^i}$s denote the standard basis vectors of $\mathbb{R}^m$. Whenever a standard basis vector $\mathbf{e^i}$ lies in an open half-space defined by the hyperplane $\mathscr{H}$ spanned by the columns of matrix $F \in \mathbb{R}^{m\times(m-1)}$, the square sub-matrix $F[\overline{m} \setminus \{i\}, \bullet]$ of F is invertible. This result is shown in the following lemma.

Lemma 3.1. *Let $F \in \mathbb{R}^{m\times(m-1)}$ and $\mathrm{rank}(F) = m-1$. Let $\mathscr{H} \subseteq \mathbb{R}^m$ be the hyperplane spanned by the column vectors of F. Let $\mathbf{n} \neq 0$ be a normal to the hyperplane $\mathscr{H}$, that is, $\langle \mathbf{n}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in \mathscr{H}$. Let $\mathbf{e^i} \in \mathbb{R}^m$ be the standard basis vectors. If $\mathbf{n}^T\mathbf{e^i} = \mathbf{n}_i \neq 0$ for some $i \in \overline{m}$ then the sub-matrix $F[\overline{m} \setminus \{i\}, \bullet]$ is invertible.*

Proof. Let the standard basis vector $\mathbf{e^i}$ be such that $\mathbf{n}^T\mathbf{e^i} = \mathbf{n}_i \neq 0$ for some $i \in \overline{m}$. Therefore, $\mathbf{e^i}$ and $\mathscr{H}$ together span the entire space $\mathbb{R}^m$. That is, the square matrix $[F|\mathbf{e^i}]$ formed by adding an additional column vector e^i to F is invertible. Thus, $\mathrm{abs}(\det([F|\mathbf{e^i}])) = \mathrm{abs}(\det(F[\overline{m} \setminus \{i\}, \bullet])) \neq 0$ and so, $F[\overline{m} \setminus \{i\}, \bullet]$ is invertible. $\square$

A generalisation of the above lemma is stated below.

Lemma 3.2. *Let $F \in \mathbb{R}^{m\times p}$, $m > p$, and $\mathrm{rank}(F) = p$. Let $\mathscr{H}^p \subseteq \mathbb{R}^m$ be the subspace spanned by the column vectors of F. Let $\mathbf{e^i} \in \mathbb{R}^m$ for $i = 1, \ldots, m$ be the standard basis vectors. If there exists $\alpha \subseteq \overline{m}$, such that, $\mathrm{span}\left\{\mathbf{e^{j_1}}, \ldots, \mathbf{e^{j_r}}\right\} \cap \mathscr{H}^p = \{0\}$, where $j_i \in \alpha$, $i = 1, \ldots, r = |\alpha|$ then $\mathrm{rank}(F[\beta, \bullet]) = \mathrm{rank}(F)$ with $\beta = \overline{m} \setminus \alpha$.*

Proof. Consider the augmented matrix $\widetilde{F} = [I[\bullet, \alpha] \mid F] \in \mathbb{R}^{m\times(p+|\alpha|)}$. Note that I is an identity matrix of order m. From the hypothesis it is clear that $\mathrm{rank}(\widetilde{F}) = p + |\alpha|$. Now it can be verified that $\mathrm{rank}(\widetilde{F}[\alpha, \bullet]) = |\alpha|$. Since the first $|\alpha|$ columns of $\widetilde{F}[\beta, \bullet]$ are all zeroes, $\mathrm{rank}(\widetilde{F}[\beta, \bullet]) = \mathrm{rank}(F[\beta, \bullet]) = \mathrm{rank}(F) = p$. $\square$

Let $D \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times(m-1)}$. Assume that $\mathrm{rank}(F) = m-1$. Consider the hyperplane $\mathscr{H}$ spanned by the columns of F. The next lemma shows that for all G, the columns of $D+FG$ cannot cross the hyperplane $\mathscr{H}$. What this implies is, that for any $i \in \overline{m}$ and G, the ith columns of D and $D+FG$ are always on the same side of the hyperplane $\mathscr{H}$.

Lemma 3.3. *Let $D \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times(m-1)}$, with $\mathrm{rank}(F) = m-1$. Let $\mathscr{H}$ be the hyperplane spanned by the columns of F. Further, let $\mathbf{n}$ be a normal vector defining $\mathscr{H}$. Then for any G the sign patterns of the row matrices $\mathbf{n}^TD$ and $\mathbf{n}^T(D+FG)$ are same.*

Proof. Since the column vectors of F span $\mathscr{H}$ therefore, $\mathbf{n}^TF = 0$. Thus, $\mathbf{n}^TD = \mathbf{n}^T(D+FG)$. Hence, the sign patterns of the rows $\mathbf{n}^TD$ and $\mathbf{n}^T(D+FG)$ remain same for all G. $\square$

In the next lemma, it is shown that whenever two vectors lie on the same open half-space formed by a hyperplane, then any one of these vectors could be transformed into a non-negative scalar multiple of the other vector by adding a suitable vector lying in the hyperplane $\mathscr{H}$.

Lemma 3.4. *Consider a hyperplane $\mathscr{H} = \{\mathbf{v} \in \mathbb{R}^m | \langle \mathbf{n}, \mathbf{v}\rangle = 0\}$ where $\mathbf{n}$ is a normal vector defining $\mathscr{H}$. Let $\mathbf{w^1}, \mathbf{w^2} \in \mathbb{R}^m$ and $\mathbf{w^1}, \mathbf{w^2} \notin \mathscr{H}$ be such that $\mathrm{sgn}(\mathbf{n}^T\mathbf{w^1}) = \mathrm{sgn}(\mathbf{n}^T\mathbf{w^2})$ then there exists $\mathbf{h^1}, \mathbf{h^2} \in \mathscr{H}$ such that $\mathbf{w^1} + \mathbf{h^1} = a\mathbf{w^2}$ and $\mathbf{w^2} + \mathbf{h^2} = b\mathbf{w^1}$ and $a, b > 0$.*

Proof. Let $\mathbf{w^1}, \mathbf{w^2} \in \mathbb{R}^m$ and $\mathbf{w^1}, \mathbf{w^2} \notin \mathscr{H}$. Assume that $\mathrm{sgn}(\mathbf{n}^T\mathbf{w^1}) = \mathrm{sgn}(\mathbf{n}^T\mathbf{w^2})$. Since $\mathbf{w^1} \notin \mathscr{H}$, the hyperplane $\mathscr{H}$ along with $\mathbf{w^1}$ spans $\mathbb{R}^m$. Therefore, there exists $\mathbf{h^1} \in \mathscr{H}$ such that $\mathbf{w^1} + \mathbf{h^1} = a\mathbf{w^2}$. Now $\mathrm{sgn}(\mathbf{n}^T\mathbf{w^1}) = \mathrm{sgn}(\mathbf{n}^T(\mathbf{w^1} + \mathbf{h^1}))$, since $\mathbf{n}^T\mathbf{h^1} = 0$. Therefore, $\mathrm{sgn}(\mathbf{n}^T\mathbf{w^1}) = \mathrm{sgn}(\mathbf{n}^T(a\mathbf{w^2})) = \mathrm{sgn}(a(\mathbf{n}^T\mathbf{w^2})) = \mathrm{sgn}(a)\,\mathrm{sgn}(\mathbf{n}^T\mathbf{w^2})$.[1] Hence, $\mathrm{sgn}(a) = \mathrm{sgn}(\mathbf{n}^T\mathbf{w^1})/\,\mathrm{sgn}(\mathbf{n}^T\mathbf{w^2}) = +1$, so $a > 0$. By using a similar argument it can shown that there exists $\mathbf{h^2} \in \mathscr{H}$ such that $\mathbf{w^2} + \mathbf{h^2} = b\mathbf{w^1}$ and $b > 0$. $\square$

[1] Note that $\mathrm{sgn}(k_1k_2) = \mathrm{sgn}(k_1)\,\mathrm{sgn}(k_2)$ for $k_1, k_2 \in \mathbb{R}$.

The following theorem states the conditions for P-matrisability of the pair (D, F) when $\text{rank}(F) = m - 1$. That is, the theorem gives the conditions for P-matrisability by hyperplane spanned by the columns of matrix F. By convention we assume that the signs of two variables do not agree if at least one of them is zero.

Theorem 3.1. *Let $D \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times(m-1)}$. Let $\text{rank}(F) = m - 1$ and $\mathscr{H}$ denote the column span of F. Let $\mathbf{n} \neq 0$ be a normal defining the hyperplane $\mathscr{H}$, that is, $\langle \mathbf{n}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in \mathscr{H}$.*

Then the matrix pair (D, F) is P-matrisable if and only if at least one position of the row vectors $\mathbf{n}^T$ and $\mathbf{n}^T D$ have the same non-zero sign pattern.

Proof. Without loss of generality, it is enough to consider only those positions where the vector $\mathbf{n}^T$ have non-zero entries. Assume that in at least one such position, the row vectors $\mathbf{n}^T$ and $\mathbf{n}^T D$ have entries with the same sign. Let $i \in \overline{m}$ be such a position. That is, the column vector $D[\bullet, i]$ of the matrix D and the standard basis vector $\mathbf{e^i} \in \mathbb{R}^m$ lie on the same side of the hyperplane $\mathscr{H}$. Then, by Lemma 3.4 there exists $a > 0$ such that $D[\bullet, i] + \mathbf{h} = a\mathbf{e^i}$ for some $\mathbf{h} \in \mathscr{H}$. This implies that there exists G such that $M = D + FG$ where $M[\bullet, i] = D[\bullet, i] + \mathbf{h} = D[\bullet, i] + FG[\bullet, i] = a\mathbf{e^i}$ for some $G[\bullet, i] \in \mathbb{R}^{m-1}$. Let $\beta = \overline{m} \setminus \{i\}$. Then, using Lemma 2.1 it is enough to show that there exists $G[\bullet, \beta]$ such that $M[\beta, \beta] = D[\beta, \beta] + F[\beta, \bullet]G[\bullet, \beta]$ is a P-matrix. By Lemma 3.1 the square matrix $F[\beta, \bullet]$ is invertible. Hence, there exists $G[\bullet, \beta]$ such that $M[\beta, \beta]$ is a P-matrix.

Conversely, assume that for all $i \in \overline{m}$ either $\text{sgn}(\langle \mathbf{n}, \mathbf{e^i}\rangle) \neq \text{sgn}(\langle \mathbf{n}, D[\bullet, i]\rangle)$ (or $\text{sgn}(\langle \mathbf{n}, \mathbf{e^i}\rangle) = \text{sgn}(\langle \mathbf{n}, D[\bullet, i]\rangle) = 0$). Geometrically, this condition is equivalent to the following possibilities: (a) whenever $\text{sgn}(\langle \mathbf{n}, \mathbf{e^i}\rangle) \neq \text{sgn}(\langle \mathbf{n}, D[\bullet, i]\rangle)$, the vectors $\mathbf{e^i}$ and $D[\bullet, i]$ lie on the opposite half spaces formed by the hyperplane $\mathscr{H}$, or one of the two vectors $\mathbf{e^i}$ or $D[\bullet, i]$ lie on the hyperplane $\mathscr{H}$ and the other lie in open half space formed by the $\mathscr{H}$ (b) both the vectors $\mathbf{e^i}$ and $D[\bullet, i]$ lie on the hyperplane $\mathscr{H}$ whenever $\text{sgn}(\langle \mathbf{n}, \mathbf{e^i}\rangle) = \text{sgn}(\langle \mathbf{n}, D[\bullet, i]\rangle) = 0$.

Suppose there exists a G such that $M = D + FG$ is a P-matrix. By Lemma 3.3 the corresponding columns of D and M lie on the same closed half space defined by hyperplane $\mathscr{H}$. This implies that either the vectors $M[\bullet, i]$ of M and standard basis vectors $\mathbf{e^i}$ lie on the opposite half spaces induced by the hyperplane $\mathscr{H}$ (when both $\langle \mathbf{n}, D[\bullet, i]\rangle \neq 0$ and $\text{sgn}(\langle \mathbf{n}, \mathbf{e^i}\rangle) \neq 0$) or, $D[\bullet, i]$ and/or $M[\bullet, i]$ lie on the hyperplane $\mathscr{H}$ (when $\langle \mathbf{n}, M[\bullet, i]\rangle = 0$ and/or $\langle \mathbf{n}, D[\bullet, i]\rangle = 0$).

Using Lemma 2.3 , for all $i = 1, 2, \ldots, m$, there exists $a_i, b_i \geq 0$ and $a_i + b_i = 1$ such that $a_i M[\bullet, i] + b_i\mathbf{e^i} \in \mathscr{H}$. Now, let M_i be the matrix obtained from M in which the first i column vectors of M has been replaced in the following manner: $M[\bullet, j]$ has been replaced by the vector $a_j M[\bullet, j] + b_j\mathbf{e^j} \in \mathscr{H}$ for all $j = 1, 2, \ldots, i$. Again applying Lemma 2.3 , matrix M_1 (whose first column vector lies on the hyperplane $\mathscr{H}$) is a P-matrix. Similarly, M_i (in which the first i column vectors lie on the hyperplane $\mathscr{H}$) is a P-matrix. Therefore, by the same argument M_m is P-matrix in which all column vectors lie on the hyperplane $\mathscr{H}$. But determinant of M_m is zero as all its m column vectors lie on the $m - 1$ dimensional hyperplane $\mathscr{H}$. This contradicts the assumption that M is a P-matrix since P-matrices are invertible. $\square$

It is clear from the above result that a decision algorithm for P-matrisability by hyperplanes is obtained by multiplying a row matrix (normal to hyperplane) with the given matrix and then comparing the sign patterns. This involves at most m^2 multiplications, $m(m - 1)$ additions, and m sign comparisons. Thus the complexity of this decision algorithm is $O(m^2)$.

We now illustrate the above theorem by a numerical example.

Example 3.1. Let

$$D_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Clearly, the matrices D_1 and D_2 are not P-matrices as $\det(D_1) = -3 < 0$ and the principal minor $D_2[1, 1] = -1 < 0$. Let $\mathscr{H} = \operatorname{im}\mathbf{f}$.The vector $\mathbf{n}^T = (1\ \ -1)$ is a normal to $\mathscr{H}$. The signs of all the components of the vector $\mathbf{n}^T$ disagree with the corresponding components of the vector $\mathbf{n}^T D_1 = (-1\ 1)$. Therefore, by Theorem 3.1 the pair $(D_1, \mathbf{f})$ is not P-matrisable. Now consider the product $\mathbf{n}^T D_2 = (1\ 1)$. The first components of the row vectors $\mathbf{n}^T$ and $\mathbf{n}^T D_2$ agree. Therefore, by Theorem 3.1 the pair $(D_2, \mathbf{f})$ is P-matrisable. Consider the feedback matrix $G = [2\ 3]$. Then,

$$D_1 + \mathbf{f}G = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $D_2 + \mathbf{f}G$ is a P-matrix.

4. Generalisations of P-matrisability

In this section we consider a generalisation of P-matrisability to matrix k-tuples. A generalisation of the LCP($\mathbf{q}$, M) is the horizontal linear complementarity problem [14–16] HLCP($\mathbf{q}$, M_1, M_2). The problem is stated as follows:

Problem 4.1. Given a vector $\mathbf{q} \in \mathbb{R}^m$, and square matrices $M_1, M_2 \in \mathbb{R}^{m\times m}$ find $\mathbf{x}, \mathbf{y} \succeq 0$ such that $M_1\mathbf{x} = \mathbf{q} + M_2\mathbf{y}$ and $\mathbf{x} \perp \mathbf{y}$.

It may be noted that the LCP($\mathbf{q}$, M) is really HLCP($\mathbf{q}$, I, M), where I is an identity matrix. A further generalisation of HLCP is the extended horizontal linear complementarity problem [15,16] EHLCP($\mathbf{q}$, $(M_j)_{(j=1)}^k$) for some $k \in \mathbb{N}$ and is defined as follows:

Problem 4.2. Given a vector $\mathbf{q} \in \mathbb{R}^m$, and square matrices $M_j \in \mathbb{R}^{m\times m}$ for $j \in \overline{k}$, find vectors $\mathbf{z^j} \in \mathbb{R}^m$ for $j \in \overline{k}$ satisfying the following conditions:

$$\begin{aligned} &M_1\mathbf{z^1} = \mathbf{q} + \sum_{j=2}^{k} M_j\mathbf{z^j}, \\ &[\mathbf{z^j}]_{j=1}^k \in \mathcal{HC}_k^m. \end{aligned}$$

The notation $[\mathbf{z^j}]_{j=1}^k \in \mathcal{HC}_k^m$ [17] denotes the constraints:

$$\begin{aligned} &\mathbf{z^1},\ \mathbf{z^k} \succeq 0, \\ &0 \preceq \mathbf{z^j} \preceq \mathbf{1} \ \text{ for } j = 2, 3, \ldots, k-1, \\ &\mathbf{z^1} \perp \mathbf{z^2}, \\ &(\mathbf{1} - \mathbf{z^j}) \perp \mathbf{z^{j+1}} \ \text{ for } j = 2, 3, \ldots, k-1, \end{aligned}$$

where the bold symbol $\mathbf{1}$ is a vector whose every entry is 1.

Note that HLCP($\mathbf{q}$, M_1, M_2) is same as EHLCP($\mathbf{q}$, $(M_j)_{(j=1)}^k$) when $k = 2$. The well-posedness of HLCP($\mathbf{q}$, M_1, M_2) and EHLCP($\mathbf{q}$, $(M_j)_{(j=1)}^k$) depends on the “column-W property” of the matrix k-tuple [18]. To define the column-W property of a matrix k-tuple, a notion of column representative matrix needs to be defined first.

Definition 4.1. A matrix M is said to be a column representative matrix of $(M_j)_{(j=1)}^k$ if $M[\bullet, i] \in \{M_j[\bullet, i] \mid j = 1, 2, \ldots, k\}$. That is, every column of the column representative matrix M is obtained by choosing from one of the corresponding columns of matrices M_j for $j = 1, 2, \ldots, k$.

Given a k-tuple of matrices $(M_j)_{(j=1)}^k$ where each $M_j \in \mathbb{R}^{m\times m}$, the set of column representative matrices will be represented by $\mathcal{C}([M_j]_{(j=1)}^k)$.

A matrix $M_\tau \in \mathcal{C}([M_j]_{(j=1)}^k)$ could be identified by $\tau = (\tau_1, \ldots, \tau_m)$ where each $\tau_i \in \bar{k}$ and $M_\tau[\bullet, i] = M_{\tau_i}[\bullet, i]$. Thus, the matrix M_τ is formed by collecting the first column of M_{τ_1} as its first column, the second column of M_{τ_2} as its second column and so on.

The column-W property of a matrix k-tuple is as defined below.

Definition 4.2. A matrix k-tuple is said to have the column-W property if the determinants of all its column representative matrices have the same sign.

It is clear that a square matrix M is a P-matrix if and only if the matrix pair (I, M) has the column-W property. The following result relates the column-W property of a matrix pair to a P-matrix.

Lemma 4.1 [18]. *A matrix pair (A, B) has the column-W property if and only if $A^{-1}B$ is a P-matrix.*

Since there is a close relationship between the well-posedness of LCP and P-matrices, one would expect that the well-posedness of an EHLCP to depend on the column-W property of a certain matrix k-tuple. This happens to be true and is established in the following lemma.

Lemma 4.2 [18]. *The EHLCP$(\mathbf{q}, (M_j)_{(j=1)}^k)$ is well-posed (that is, has a unique solution for every $\mathbf{q} \in \mathbb{R}^m$) if and only if the matrix k-tuple $(M_j)_{(j=1)}^k$ has the column-W property.*

4.1. Motivation for W-matrisability problem

Similar to LCP many applications can be modelled using EHLCP. One such application is in modelling piecewise linear systems (PLS). Many PLSs (sometimes called piecewise affine systems) can be transformed into a form in which the dynamic versions of HLCP or EHLCP appear as constraints. These systems arise when piecewise linear characteristic constraints are imposed on certain variables of linear dynamical systems (see [9,17] for more details). These PLSs, in turn, are generalised versions of LCSs. Mathematically, such PLSs are represented using the following relations [17]:

$$\dot{\mathbf{x}}(t) = \mathbf{q^1} + A\mathbf{x}(t) + N_1\mathbf{y^1}(t) + \ldots + N_k\mathbf{y^k}(t) + E\mathbf{w}(t), \tag{3a}$$

$$D_1\mathbf{y^1}(t) = \mathbf{q^2} + C\mathbf{x}(t) + D_2\mathbf{y^2}(t) + \ldots + D_k\mathbf{y^k}(t) + F\mathbf{w}(t), \tag{3b}$$

$$[\mathbf{y^j}]_{j=1}^k \in \mathcal{HC}_k^m, \tag{3c}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{y^j}(t) \in \mathbb{R}^m$, for $j \in \bar{k}$, $\mathbf{q^1} \in \mathbb{R}^n$, $\mathbf{q^2} \in \mathbb{R}^m$, $\mathbf{w}(t) \in \mathbb{R}^p$, and matrices $(D_j)_{j=1}^k$, $(N_j)_{j=1}^k$, A, C, E and F are of compatible dimensions. The vector $\mathbf{w}(t)$ denotes the input variables of the PLS. Note that following the terminology of LCS one could call the PLS defined in (3) as extended horizontal linear complementarity systems (EHLCS). But in this paper the term PLS is used in the sense of EHLCS.

A PLS is said to be well-posed if there exists a unique solution for every initial state and each admissible input. Hence under the condition that $(D_j)_{j=1}^k$ has the column-W property the PLS (3) is well-posed. It is clear that the application of the feedback $\mathbf{w}(t) = -G_1\mathbf{y^1}(t) + \sum_{j=2}^k G_j\mathbf{y}^j + \mathbf{v}(t)$ results in the PLS (3) being modified to:

$$\dot{\mathbf{x}}(t) = \mathbf{q^1} + A\mathbf{x}(t) + (N_1 - EG_1)\mathbf{y^1}(t) + \sum_{j=2}^k (N_j + EG_j)\mathbf{y^j}(t) + E\mathbf{v}(t), \tag{4a}$$

$$(D_1 + FG_1)\mathbf{y^1}(t) = \mathbf{q^2} + C\mathbf{x}(t) + \sum_{j=2}^k (D_j + FG_j)\mathbf{y^j}(t) + F\mathbf{v}(t), \tag{4b}$$

$$[\mathbf{y^j}]_{j=1}^k \in \mathcal{HC}_k^m. \tag{4c}$$

Thus if the matrix k-tuple $(D_j + FG_j)_{(j=1)}^k$ has the column-W property then the EHLCP($\mathbf{q^2} + C\mathbf{x}(t) + F\mathbf{v}(t)$, $(D_j + FG_j)_{(j=1)}^k$) is well-posed. This ensures that the feedback PLS (4) is also well-posed. Now, if one could choose matrices G_j such that $(D_j + FG_j)_{(j=1)}^k$ has the column-W property then one would have regularised the PLS (3).

Motivated by the PLS regularisation problem, the problem of generating the matrix k-tuples with column-W property from a given $(k+1)$-tuple of matrices is considered in this paper. The problem is: Given a matrix k-tuple $(D_j)_{(j=1)}^k$ and a matrix F, determine the conditions for which there exists matrices $G_j \in \mathbb{R}^{p\times m}$ such that the matrix k-tuple $(D_j + FG_j)_{(j=1)}^k$ has the column-W property. The $(k+1)$-tuple $((D_j)_{(j=1)}^k, F)$ is said to be column-W-matrisable if there exists matrices $G_j \in \mathbb{R}^{p\times m}$ such that the matrix k-tuple $(D_j + FG_j)_{(j=1)}^k$ has the column-W property.

Whenever the matrix $(k+1)$-tuple $((D_j)_{(j=1)}^k, F)$ is column-W-matrisable the matrix k-tuple $(D_j)_{(j=1)}^k$ is said to be column-W-matrisable by the subspace spanned by columns of F.

The complexity of the algorithms for checking column-W property is at least co-NP-complete. This is because it has been shown that checking if a matrix is a P-matrix is co-NP-complete [19]. In the subsequent subsections a decision algorithm is developed for checking column-W-matrisability by a hyperplane having complexity $O(km^2)$.

4.2. Properties of Matrix k-tuples satisfying column-W condition

In this section some properties of P-matrices are extended to the matrix k-tuples that satisfy column-W condition. First, the notion of permutation similarity of a matrix is generalised to matrix pairs. If P is a permutation matrix then $(P^{-1}AP, P^{-1}BP)$ is said to be permutationally similar to (A, B). It can be verified that if P is a permutation matrix then the column representative matrices of $(P^{-1}AP, P^{-1}BP)$ are permutationally similar to column representative matrices of (A, B). Therefore, the pair (A, B) has the column-W property if and only if $(P^{-1}AP, P^{-1}BP)$ has the column-W property, when P is a permutation matrix.

The block diagonal form of a matrix is generalised in the following way: A matrix pair (A, B) is said to be in upper column-tuple-block diagonal form if there exists an $l \in \overline{m}$ such that $B[\bullet, i] = \sum_{j=1}^{l} a_{i,j}A[\bullet, j]$ for $a_{i,j} \in \mathbb{R}$, $i \leq l$ and remaining columns of B are arbitrary. In fact, if A is invertible, the matrix $A^{-1}B$ is in the upper block diagonal form. Similarly, a lower column-tuple-block diagonal form for a matrix pair (A, B) could also be defined. In general, it could be said that a matrix pair (A, B) is in column-tuple-block diagonal form if it is either in upper or lower column-tuple-block form.

Lemma 4.3. *Consider a matrix pair (A, B) where $B[\bullet, i] = A[\bullet, i]$ for all $i \in \alpha$ where $\alpha \subseteq \overline{m}$, and $B[\bullet, l] = A[\bullet, l] + \sum_{i\in\alpha} a_{i,l}A[\bullet, i]$ for $a_{i,l} \in \mathbb{R}$, and for all $l \notin \alpha$. Then the matrix pair (A, B) has the column-W property if and only if A is invertible.*

Proof. A matrix pair (A, B) where $B[\bullet, i] = A[\bullet, i]$ for all $i \in \alpha$ where $\alpha \subseteq \overline{m}$, and $B[\bullet, l] = A[\bullet, l] + \sum_{i\in\alpha} a_{i,l}A[\bullet, i]$ for $a_{i,l} \in \mathbb{R}$, and $l \notin \alpha$ is clearly permutation-similar to tuple-block diagonal form. It can be verified that the determinant of every column representative matrix of this pair (A, B) is same as that of A. Therefore, if A is invertible then this pair (A, B) has the column-W property.

Conversely, if (A, B) has the column-W property then clearly A is one of the column representative matrices. That is, its determinant is non-zero and hence A is invertible. □

The notion of column-tuple-block diagonal form of a matrix pair can also be extended to matrix k-tuples $(M_j)_{(j=1)}^k$ as follows: A matrix k-tuple $(M_j)_{(j=1)}^k$ is said to be in upper column-tuple-block diagonal form, if there exists an $l \in \overline{m}$ such that $M_j[\bullet, i] = \sum_{r=1}^{l} a_{i,j,r}M_1[\bullet, r]$ for $a_{i,j,r} \in \mathbb{R}$, $i \leq l$, $2 \leq j \in \bar{k}$ and remaining columns of M_j are arbitrary. Similarly, one could define a lower column-tuple-block diagonal form for a matrix k-tuple $(M_j)_{(j=1)}^k$. In general, a matrix k-tuple $(M_j)_{(j=1)}^k$ is said to be in column-tuple-block diagonal form if it is either in upper or lower column-tuple-block form. It can be verified that for $k = 2$ this definition of column-tuple-block diagonal form reduces to column-tuple-block diagonal form for the matrix pair (M_1, M_2).

The concept of permutation-similarity can also be naturally extended to matrix k-tuples $(M_j)_{(j=1)}^k$. A matrix k-tuple $(P^{-1}M_jP)_{(j=1)}^k$ is said to be permutationally similar to $(M_j)_{(j=1)}^k$ if P is a permutation matrix. It can be verified that if P is a permutation matrix, then the column representative matrices of $(P^{-1}M_jP)_{(j=1)}^k$ are permutationally similar to column representative matrices of $(M_j)_{(j=1)}^k$. Therefore, matrix k-tuple $(M_j)_{(j=1)}^k$ has the column-W property if and only if $(P^{-1}M_jP)_{(j=1)}^k$ has the column-W property, where P is a permutation matrix.

Corresponding to Lemma 4.3 is the following result for column-tuple-block diagonal matrix k-tuples.

Lemma 4.4. *A matrix k-tuple $(M_j)_{(j=1)}^k$ where $M_j[\bullet, i] = M_1[\bullet, i]$ for $i \in \alpha$, $\alpha \subseteq \overline{m}$ and $M_j[\bullet, l] = M_1[\bullet, l] + \sum_{i\in\alpha} a_{j,i}M_1[\bullet, i]$ for $a_{j,i} \in \mathbb{R}$, $l \in \beta$ with $\beta = \overline{m} \setminus \alpha$ has the column-W property if and only if M_1 is invertible.*

Proof. It can be verified that a given matrix k-tuple $(M_j)_{(j=1)}^k$ satisfying the hypothesis is permutationally similar to column-tuple-block diagonal form. The result now follows from the fact that determinant of every column representative matrix of this matrix k-tuple is same as that of M_1. $\square$

The following set of lemmata gives generalisations of the column hull property of the P-matrices to matrix k-tuples.

Lemma 4.5. *The matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ has the column-W property if and only if $\sum_{j=1}^k M_jD_j$ is invertible for all diagonal matrices $D_j \geq 0$ and $\sum_{j=1}^k D_j$ is a diagonal matrix with non-zero diagonals.*

Proof. (**If**): Assume that the matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ has the column-W property. One needs to show that $\sum_{j=1}^k M_jD_j$ is invertible for all diagonal matrices $D_j \geq 0$ (with non-negative entries) and $\sum_{j=1}^k D_j$ is a diagonal matrix with positive diagonal entries. That is, to show that $\det(\sum_{j=1}^k M_jD_j) \neq 0$. Due to multi-linearity of the determinant function one has $\det(\sum_{j=1}^k M_jD_j) = \sum_\tau \det(M_\tau D_\tau)$, where $D_\tau \in \mathcal{C}((D_j)_{j=1}^k)$, $M_\tau \in \mathcal{C}((M_j)_{j=1}^k)$ and $\tau = (\tau_1, \ldots, \tau_m)$, $\tau_i \in \overline{k}$, $D_\tau[\bullet, i] = D_{\tau_i}[\bullet, i]$ and $M_\tau[\bullet, i] = M_{\tau_i}[\bullet, i]$ for $i \in \overline{m}$. Without loss of generality, it can be assumed that $\det(M_\tau) > 0$ for all τ, since the matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ has the column-W property. Since D_js are diagonal matrices with non-negative entries $\det(D_\tau) \geq 0$. Thus, the product $M_\tau D_\tau$ has non-negative determinant for all τ. Hence, $\sum_\tau \det(M_\tau D_\tau) \geq 0$. But since $\sum_{j=1}^k D_j$ is a diagonal matrix with positive diagonal entries, there exists m-tuple ω such that $D_\omega \in \mathcal{C}((D_j)_{j=1}^k)$ and $\det(D_\omega) > 0$. This means $\det(M_\omega D_\omega) > 0$. Consequently, $\sum_\tau \det(M_\tau D_\tau) > 0$ and hence $\det(\sum_{j=1}^k M_jD_j) > 0$. This means that the matrix $\sum_{j=1}^k M_jD_j$ is invertible for all D_j's satisfying the hypothesis.

(**Only if**): Assume that $\sum_{j=1}^k M_jD_j$ is invertible for all diagonal matrices $D_j \geq 0$ (with non-negative entries) and $\sum_{j=1}^k D_j$ is a diagonal matrix with positive diagonal entries. Due to continuity, $\det(\sum_{j=1}^k M_jD_j)$ is either positive or negative for all D_js satisfying the hypothesis. Now to show that the matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ has the column-W property, it is required to show that for all $M_\tau \in \mathcal{C}((M_j)_{j=1}^k)$ and $\tau = (\tau_1, \ldots, \tau_m)$, $\tau_i \in \overline{k}$, $\det(M_\tau)$ are consistently positive or negative. Now, for a given τ choose T_js for $j \in \overline{k}$ such that $T_{\tau_i}[\bullet, i] = I[\bullet, i]$ and the remaining columns to be zero. Then, clearly T_js are diagonal matrices satisfying $T_j \geq 0$ and $\sum_{j=1}^k T_j = I$ is a diagonal matrix with positive entries. It can be verified that $\sum_{j=1}^k M_jT_j = M_\tau$. Hence $\det(M_\tau)$ is either consistently positive or negative depending on $\det(\sum_{j=1}^k M_jT_j)$. $\square$

As a corollary of the extended column hull property, the column hull property for matrix k-tuple is stated as follows:

Lemma 4.6. *The matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ has the column-W property if and only if $\sum_{j=1}^k M_jD_j$ is invertible for all $D_j \in [0, I]$ and $\sum_{j=1}^k D_j = I$.*

Proof. The proof follows from the extended column hull property above. □

4.3. W-matrisability of matrix pairs by hyperplane

The following result is the generalisation of the P-matrisability problem by hyperplane to the column-W-matrisability problem of a matrix pair by hyperplane. Recall the convention that the signs of two variables do not agree if one of them is zero.

Theorem 4.3. *Consider a matrix pair $(A, B) \in (\mathbb{R}^{m\times m})^2$, $F \in \mathbb{R}^{m\times(m-1)}$. Let $\operatorname{rank}(F) = m-1$ and $\mathscr{H}$ denote the hyperplane generated by the columns of F. Let $\mathbf{n} \in \mathbb{R}^m$ be a normal to $\mathscr{H}$. Assume that matrix A is invertible.*

Then there exists a matrix G such that the matrix pair $(A, B + FG)$ has the column-W property if and only if the non-zero sign patterns of vectors $\mathbf{n}^TA$ and $\mathbf{n}^TB$ agree on at least one position.

Proof. Suppose there is at least one matching sign in the row vectors $\mathbf{n}^TA$ and $\mathbf{n}^TB$. Let $i \in \overline{m}$ be a position such that $\operatorname{sgn}(\mathbf{n}^TA[\bullet, i]) = \operatorname{sgn}(\mathbf{n}^TB[\bullet, i]) \neq 0$. This means that the column vectors $A[\bullet, i]$ and $B[\bullet, i]$ are on the same open half space formed by the hyperplane $\mathscr{H}$. Therefore, by Lemma 3.4 there exists $\mathbf{h}^i \in \mathscr{H}$ such that $B[\bullet, i] + \mathbf{h}^i = a_iA[\bullet, i]$ with $a_i > 0$. Further, since $A[\bullet, i] \notin \mathscr{H}$, the column vector $A[\bullet, i]$ and $\mathscr{H}$ span the whole space. So, there exists $\mathbf{h}^l \in \mathscr{H}$ for every $l \in \overline{m}\setminus\{i\}$ such that $B[\bullet, l] - A[\bullet, l] = a_lA[\bullet, i] - \mathbf{h}^l$. That is, $B[\bullet, l] + \mathbf{h}^l = a_lA[\bullet, i] + A[\bullet, l]$. Therefore, there exists G such that $(B + FG)[\bullet, i] = a_iA[\bullet, i]$ and $(B + FG)[\bullet, l] = a_lA[\bullet, i] + A[\bullet, l]$ for $l \in \overline{m}\setminus\{i\}$. Hence by using Lemma 4.3 it follows that the matrix pair $(A, B + FG)$ has the column-W property.

Conversely, let the sign patterns of the corresponding entries of the row vectors $\mathbf{n}^TA$ and $\mathbf{n}^TB$ either disagree or be zero. This means that the column vectors $A[\bullet, i]$ and $B[\bullet, i]$ are either on opposite half spaces of the hyperplane $\mathscr{H}$ or at least one of them lie on $\mathscr{H}$ for $i \in \overline{m}$. Assume for contradiction that there exists G such that the matrix pair $(A, B+FG)$ has the column-W property. So, by Lemma 4.6 the matrix $AD_1 + (B+FG)D_2$ is invertible for all diagonal matrices $D_1, D_2 \in [0, I]$. That is, the columns of $AD_1 + (B + FG)D_2$ are in the convex hull of the corresponding columns of A and $B + FG$. Lemma 3.3 implies that the sign pattern of $\mathbf{n}^T(B+FG)$ is same as $\mathbf{n}^TB$. Which means that the columns of $B+FG$ do not cross over to other side of the hyperplane $\mathscr{H}$. That is, the corresponding columns of A and $B + FG$ are on opposite sides of the hyperplane $\mathscr{H}$ or at least one of them lie on $\mathscr{H}$. Thus, there exists some $Q_1, Q_2 \in [0, I]$ such that all the columns of $AQ_1 + (B + FG)Q_2$ lie on hyperplane $\mathscr{H}$. But this implies that $AQ_1 + (B + FG)Q_2$ is not invertible which is a contradiction to assumption that the matrix pair $(A, B + FG)$ has the column-W property. □

It is clear from Theorem 4.3 that an algorithm for checking column-W-matrisability of a matrix pair (A, B) by a hyperplane can be obtained by multiplying the matrices A and B by the normal to $\mathscr{H}$ and comparing the signs of the resultant vectors. This requires at most $2m^2$ multiplications, $2m(m-1)$ additions and m sign comparisons. Thus the complexity of this algorithm is $O(m^2)$.

We now illustrate the above result with an example.

Example 4.1. Let

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

It can be verified that the matrix pairs (A, B_1) and (A, B_2) do not have the column-W property. Let $\mathscr{H} = \operatorname{im} \mathbf{f}$. A normal to $\mathscr{H}$ is given by $\mathbf{n}^T = (1\ 1)$. Now the products $\mathbf{n}^T A = (-1\ \ -3)$ and $\mathbf{n}^T B_1 = (3\ 1)$. All the signs of the corresponding components of the vectors $\mathbf{n}^T A$ and $\mathbf{n}^T B_1$ disagree. Therefore, by Theorem 4.3 the triplet $(A, B_1, \mathbf{f})$ is not W-matrisable. On the other hand, the signs of the first components of the vectors $\mathbf{n}^T A = (-1\ \ -3)$ and $\mathbf{n}^T B_2 = (-3\ 1)$, are the same. Therefore, by Theorem 4.3 the triplet $(A, B_2, \mathbf{f})$ is W-matrisable. Consider the feedback $G = [4\ \ -1]$. Then,

$$B_2 + \mathbf{f}G = \begin{bmatrix} -6 & 3 \\ 3 & -2 \end{bmatrix}.$$

It can be verified that

$$B_2 + \mathbf{f}G = \left(3A[\bullet, 1] \qquad \frac{-8}{5}A[\bullet, 1] + \frac{1}{5}A[\bullet, 2]\right).$$

Since matrix A is invertible, the determinants of all column representative matrices of the pair $(A, B_2 + \mathbf{f}G)$ are all non-zero and they all have the same sign. Hence the pair $(A, B_2 + \mathbf{f}G)$ has the column-W property.

4.4. W-matrisability problems for matrix k-tuples

It is clear from the definition of the column-W property that it depends solely on the column representative matrices obtained from the set of columns of a given set of matrices. Therefore, for determining the column-W property of a matrix k-tuple, it is enough to consider a collection of m sets of vectors obtained from the columns of the matrix k-tuple.

Consider the sets $\mathcal{M}^i$, $i \in \overline{m}$, where $\mathcal{M}^i$ contains the ith columns of the matrices M_j of a matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$. A column representative matrix R of the matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ is obtained by choosing the ith column of R from $\mathcal{M}^i$ for $i \in \overline{m}$. Thus as far as determining the column-W property of a matrix k-tuple is concerned, the collection $(\mathcal{M}^i \mid i \in \overline{m})$ could be thought of as representing the matrix k-tuple. Conversely, any collection $(\mathcal{M}^i \mid i \in \overline{m})$ can be thought of as an appropriate matrix k-tuple, where the k is the cardinality of the set $\mathcal{M}^i$ containing maximum number of elements. Hence instead of working with the matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ we will work with the collection $(\mathcal{M}^i \mid i \in \overline{m})$.

So, consider $\mathcal{M}$, the collection of sets defined as follows:

$$\mathcal{M} = (\mathcal{M}^i \mid i \in \overline{m}), \tag{5a}$$

$$\mathcal{M}^i = \{\mathbf{m}^{i,j} \in \mathbb{R}^m \mid j \in \overline{m_i} \text{ and } m_i \in \mathbb{N}\} \text{ for } i \in \overline{m}. \tag{5b}$$

Note that the cardinality of $\mathcal{M}$ is m and the cardinality of $\mathcal{M}^i$ is m_i for $i \in \overline{m}$. A representative matrix of such a collection is obtained by choosing the ith column[2] of R from $\mathcal{M}^i$ for $i \in \overline{m}$. Let $\mathcal{R}_{\mathcal{M}}$ denote the set of all representative matrices of the collection $\mathcal{M}$. Then the collection $\mathcal{M}$ is said to have the W-property if the signs of the determinants of every representative matrix $R \in \mathcal{R}_{\mathcal{M}}$ (of the collection $\mathcal{M}$) are all positive or all negative. The collection $\mathcal{M}$ is said to be degenerate, if there exists at least one $R \in \mathcal{R}_{\mathcal{M}}$ such that $\det(R) = 0$. It is clear that every collection $\mathcal{M}$ having the W-property is non-degenerate. Clearly, checking the column-W property of a matrix k-tuple $(M_j)_{j=1}^k$ is equivalent to checking the W-property of an appropriate collection $\mathcal{M}$.

Consider now a finite set of vectors $\mathcal{S} \subset \mathbb{R}^m$. Every set $\mathcal{S}$ defines a polyhedral cone $\mathrm{PosCone}(\mathcal{S}) = \sum_{s\in\mathcal{S}} a_s s$ and $a_s \geq 0$. A cone $\mathcal{C}$ is said to be proper if

[2] Note that one could have defined a representative matrix R by choosing the ith row of R from $\mathcal{M}^i$ for $i \in \overline{m}$. However, this amounts to working with just the transposes of the original representative matrices. Also, most of the results in this paper, depend on evaluating determinants. So, it is enough to just work with the definition using columns.

(a) $0 \in \mathcal{C}$,
(b) $\mathcal{C}$ is solid, that is, the interior of $\mathcal{C}$ is non-empty, that is, every set which is open relative to $\mathcal{C}$ is also open in $\mathbb{R}^m$,
(c) there exists no subspace $\mathscr{V} \subseteq \mathbb{R}^m$ such that $\mathscr{V} \subseteq \mathcal{C}$.

The notion of independence of vectors of a vector space can be extended to cones as follows:

Definition 4.3. A set $\mathcal{S}$ of vectors is said to be polyhedrally independent if $|\mathcal{S}|$ is the minimum set of vectors required to generate the cone $\mathrm{PosCone}(\mathcal{S})$.

Corresponding to the column hull property of the matrix k-tuples the generalised hull property for a collection $\mathcal{M}$ is stated as follows:

Lemma 4.7. *A collection $\mathcal{M}$ has the W-property if and only if invertibility is guaranteed for each square matrix R obtained by choosing the columns such that $R[\bullet, i] \in \mathrm{PosCone}(\mathcal{M}^i)$ and $i \in \overline{m}$.*

Proof. Follows from the equivalence of an m-tuple $\mathcal{M}$ to some matrix k-tuple $(M_j)_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ and the extended column hull property Lemma 4.5 for matrix k-tuples. □

From the above lemma and the definition of W-property (of a collection) it is clear that the 'rays' of the different cones play an important role in determining the W-property of any collection. So, a notion of equivalence of collections can be defined as follows:

Definition 4.4. Given a collection of sets $\mathcal{M}$, another collection of sets $\mathcal{N}$ is said to be polyhedrally equivalent if the polyhedral cones $\mathrm{PosCone}(\mathcal{M}^i) = \mathrm{PosCone}(\mathcal{N}^i)$ for $i \in \overline{m}$.

If in addition, the sets $\mathcal{N}^i$ are polyhedrally independent, then $\mathcal{N}$ is said to be a minimal representation of $\mathcal{M}$. Note that minimal representation of $\mathcal{M}$ is not unique but the cardinality of the minimal representations is invariant. If the vectors of a minimal representation are normalised to unit length, then the minimal representation so obtained is unique. These unit vectors define the "rays" of the polyhedral cone $\mathrm{PosCone}(\mathcal{M}^i)$. The computational advantage of minimal representation of a collection is that, it does not contain any redundancy as far as the W-property of the original collection is concerned.

The following lemma shows that the W-property of a collection $\mathcal{M}$ is invariant for every other collection $\mathcal{N}$ which is polyhedrally equivalent to the collection $\mathcal{M}$.

Lemma 4.8. *Consider a collection $\mathcal{M}$ such that $\mathbf{0} \notin \mathcal{M}^i$, for $i \in \overline{m}$. Let $\mathcal{N}$ be another collection, which is polyhedrally equivalent to $\mathcal{M}$. Then $\mathcal{M}$ has the column-W property if and only if $\mathcal{N}$ has the column-W property.*

Proof. This result follows Lemma 4.7 . □

Due to the above result whenever a collection $\mathcal{M}$ has the W-property, choosing a minimal representation of $\mathcal{M}$ is enough for the W-property.

Consider a collection $\mathcal{M}$. A collection $\mathcal{N}$ is said to be a (non-degenerate) sub-collection of $\mathcal{M}$ if $\mathcal{N}^i \neq \emptyset$ and $\mathcal{N}^i \subseteq \mathcal{M}^i$ for $i \in \overline{m}$. Clearly, if the collection $\mathcal{M}$ has the W-property then the sub-collection $\mathcal{N}$ also has the W-property. Further, it can be verified that the collection $\mathcal{M}$ has the W-property if and only if every sub-collection $\mathcal{N}$ of $\mathcal{M}$ has the W-property.

Consider a matrix k-tuple $((M_j)_{j=1}^k) \in (\mathbb{R}^{m\times m})^k$ and a collection $\mathcal{M}$ that comes from this. Let $\mathscr{L} \subseteq \mathbb{R}^m$ be any subspace. The pair $(\mathcal{M}, \mathscr{L})$ is said to be W-matrisable (or alternatively, $\mathcal{M}$ is said to be W-matrisable by a subspace $\mathscr{L}$), if there exists a collection $\widetilde{\mathcal{M}} \in \mathcal{M} + \mathscr{L}$, such that, $\widetilde{\mathcal{M}}$ has the W-property. The collection $\widetilde{\mathcal{M}} \in \mathcal{M} + \mathscr{L}$ is defined as follows:

$$\widetilde{\mathcal{M}} = \{\widetilde{\mathcal{M}^i} \mid i \in \overline{m}\},$$

$$\widetilde{\mathcal{M}^i} = \{\mathbf{m}^{i,j} + \ell^{i,j} \ \mid \ \mathbf{m}^{i,j} \in \mathcal{M}^i, \ \ell^{i,j} \in \mathscr{L}, \ \ |\widetilde{\mathcal{M}^i}| = m_i \ \text{ and } \ j \in \overline{m_i}\ \} \ \forall\, i \in \overline{m}.$$

It is clear that if $m_i = m$ for all $i \in \overline{m}$, the collection $\widetilde{\mathcal{M}}$ represents some matrix k-tuple of the form $(M_j + LG_j)_{j=1}^k$, where the columns of L spans the subspace $\mathscr{L}$ and G_js depends on the choice of $\ell^{i,j}$ in the collection $\widetilde{\mathcal{M}}$ above. Similarly, when one is given a matrix k-tuple $(M_j + LG_j)_{j=1}^k$ a collection $\widetilde{\mathcal{M}}$ like that defined above can be obtained, where $\mathscr{L}$ spans the column space of L. The following result can be easily verified:

Lemma 4.9. *The pair $(\mathcal{M}, \mathscr{L})$ is W-matrisable if and only if the pair $(\widetilde{\mathcal{M}}, \mathscr{L})$ for all $\widetilde{\mathcal{M}} \in \mathcal{M} + \mathscr{L}$ is W-matrisable.*

Proof. Follows from the definition of $\mathcal{M} + \mathscr{L}$. $\square$

The complexity of the decision algorithm for W-property of $\mathcal{M}$ is the same as that for checking column-W property of matrix k-tuples which is co-NP-complete. In the next subsection a algorithm for checking W-matrisability by hyperplanes which has a complexity of $O(km^2)$ is developed.

4.5. W-matrisability by hyperplanes

The main result in this subsection is on the W-matrisability problem by hyperplanes for a collection $\mathcal{M}$. This will be used to derive the W-matrisability by hyperplanes of matrix k-tuples. The following lemma on the properties of independence of vectors with respect to a hyperplane is used in the main result.

Lemma 4.10. *Let $\mathscr{H}$ be a hyperplane in $\mathbb{R}^m$ defined by a unit normal vector $\mathbf{n}$. Let $\mathbf{v^1} \notin \mathscr{H}$. Then for every set of vectors $\{\mathbf{v^2}, \ldots, \mathbf{v^m}\} \subseteq \mathbb{R}^m$ there exists vectors $\mathbf{h^2}, \ldots, \mathbf{h^m} \in \mathscr{H}$, such that the set $\{\mathbf{v^1}, \mathbf{v^2} + \mathbf{h^2}, \ldots, \mathbf{v^m} + \mathbf{h^m}\}$ is linearly independent.*

Proof. Obvious. $\square$

The W-matrisability problem by a hyperplane $\mathscr{H}$ and collection $\mathcal{M}$ is stated next:

Theorem 4.4. *Let $\mathcal{M}$ be defined as in (5). Let $\mathscr{H}$ be a hyperplane in $\mathbb{R}^m$ defined by a unit normal vector $\mathbf{n}$. The pair $(\mathcal{M}, \mathscr{H})$ is W-matrisable if and only if there exists an $i \in \overline{m}$ such that $\operatorname{sgn}(\langle \mathbf{n}, \mathbf{m}^{i,j} \rangle) > 0$ for all $j \in \overline{m_i}$ or $\operatorname{sgn}(\langle \mathbf{n}, \mathbf{m}^{i,j} \rangle) < 0$ for all $j \in \overline{m_i}$.*

Proof. Assume that there exists an $i \in \overline{m}$ such that $\operatorname{sgn}(\langle \mathbf{n}, \mathbf{m}^{i,j} \rangle) > 0$ for all $j \in \overline{m_i}$ or $\operatorname{sgn}(\langle \mathbf{n}, \mathbf{m}^{i,j} \rangle) < 0$ for all $j \in \overline{m_i}$. This means that all the vectors in the set $\mathcal{M}^i$ are on the same open half space defined by the hyperplane $\mathscr{H}$. Then the following observations can be made:

1. Let $\mathbf{w} \in \mathcal{M}^i$. Then by Lemma 3.4 for all $\mathbf{v} \in \mathcal{M}^i$ there exists $a_v > 0$ and $\mathbf{h} \in \mathscr{H}$ such that $\mathbf{v} + \mathbf{h} = a_v \mathbf{w}$.
2. Choose $\mathbf{v}^j \in \mathcal{M}^j$ for $j \in \overline{m} \setminus \{i\}$. Since $\mathbf{w} \notin \mathscr{H}$, from Lemma 4.10 there exists vectors $\mathbf{h^j} \in \mathscr{H}$ such that the set $\{\mathbf{w}, \mathbf{v}^2 + \mathbf{h}^2, \ldots, \mathbf{v}^m + \mathbf{h}^m\}$ is linearly independent.
3. Since $\mathbf{w} \notin \mathscr{H}$, for every $\mathbf{v} \in \mathcal{M}^j$ and $\mathbf{v} \neq \mathbf{v}^j$ there exists $\mathbf{h^{v,j}} \in \mathscr{H}$ and $a_{\mathbf{v},j} \in \mathbb{R}$ such that $\mathbf{h^{v,j}} - a_{\mathbf{v},j}\mathbf{w} = \mathbf{v}^j + \mathbf{h}^j - \mathbf{v}$ for $j \in \overline{m} \setminus \{i\}$. That is, $\mathbf{v} + \mathbf{h^{v,j}} = a_{\mathbf{v},j}\mathbf{w} + \mathbf{v}^j + \mathbf{h}^j$ for $j \in \overline{m} \setminus \{i\}$.
4. Now consider the collection $\mathcal{N}$ such that $\mathcal{N}^i = \{\mathbf{w}\}$ and

$$\mathcal{N}^j = \left\{ a_{\mathbf{v},j}\mathbf{w} + \mathbf{v}^j + \mathbf{h}^j \mid \mathbf{v} \in \mathcal{M}^j \text{ and } \mathbf{v} \neq \mathbf{v}^j \right\} \cup \{\mathbf{v}^j + \mathbf{h}^j\}.$$

Clearly the collection $\mathcal{N}$ is polyhedrally equivalent to some collection from $\mathcal{M} + \mathscr{H}$. Hence by Lemma 4.8 and Lemma 4.9 , it is enough to show that the pair $(\mathcal{N}, \mathscr{H})$ is W-matrisable.

It is claimed that the collection $\mathcal{N}$ has the W-property. Let R be a matrix such that $R[\bullet, i] = \mathbf{w}$ and $R[\bullet, j] = \mathbf{v}^j + \mathbf{h}^j$ for $j \in \overline{m} \setminus \{i\}$. Clearly R is a representative matrix of the collection $\mathcal{N}$ and $\det(R) \neq 0$.

Now, consider a typical representative matrix N of $\mathcal{N}$. The lth column of N is of the form $N[\bullet, l] = a_l\mathbf{w} + R[\bullet, l]$ for some $a_l \in \mathbb{R}$ and $N[\bullet, i] = \mathbf{w}$. Since the determinant function is multi-linear, the determinant is unchanged when one adds or subtracts a multiple of other columns to a given column. Now for every $l \in \overline{m} \setminus \{i\}$ subtract a_l times the ith column $N[\bullet, i]$ from $N[\bullet, l]$. This operation results in a matrix which is same as R. Therefore, the determinant of every representative matrix of $\mathcal{N}$ is same as that of determinant of R. Hence the collection $\mathcal{N}$ has the W-property. Therefore, the pair $(\mathcal{M}, \mathscr{H})$ is W-matrisable.

Conversely, suppose the assumptions in the hypothesis are not satisfied. Assume for contradiction that the pair $(\mathcal{M}, \mathscr{H})$ is W-matrisable. Let $\mathcal{N}$ represent a collection obtained from $\mathcal{M}$ by the action of $\mathscr{H}$ such that $\mathcal{N}$ has the W-property. By Lemma 3.3 the hypothesis is still not satisfied for $\mathcal{N}$. That is, for no $i \in \overline{m}$ the conditions $\operatorname{sgn}(\langle \mathbf{n}, \mathbf{n}^{i,j}\rangle) > 0$ or $\operatorname{sgn}(\langle \mathbf{n}, \mathbf{n}^{i,j}\rangle) < 0$ for all $j \in \overline{n_i}$ and $\mathbf{n}^{i,j} \in \mathcal{N}^i$ are consistently satisfied. Geometrically, this means that there exists no $i \in \overline{m}$ such that all the vectors in $\mathcal{N}^i$ are on the same open half plane defined by $\mathscr{H}$. Therefore, for every $i \in \overline{m}$ the cone $\operatorname{PosCone}(\mathcal{N}^i)$ has a non trivial intersection with the hyperplane $\mathscr{H}$. Consider a matrix R such that $R[\bullet, i] \in \operatorname{PosCone}(\mathcal{N}^i) \cap \mathscr{H}$ for $i \in \overline{m}$. This R is not invertible because, $\det(R) = 0$. But this is a contradiction to Lemma 4.7 . Hence the pair $(\mathcal{M}, \mathscr{H})$ is not W-matrisable. $\square$

It is clear from the above result that an algorithm for checking W-matrisability by hyperplanes and hence for column-W-matrisability of matrix k-tuples is obtained by multiplying a row matrix (normal to hyperplane) with the matrix k-tuple and then comparing the sign patterns. This involves at most km^2 multiplications, $km(m-1)$ additions, and $k(m)$ sign comparisons. Therefore, the complexity of this decision algorithm for W-matrisability is $O(km^2)$.

The above result is now illustrated by an example.

Example 4.2. Consider the collection $\mathcal{M}$ such that

$$\mathcal{M}^1 = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{M}^2 = \left\{ \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$$\mathcal{M} = \left\{ \mathcal{M}^1, \mathcal{M}^2 \right\}, \quad \mathbf{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathscr{H} = \operatorname{span}\{\mathbf{f}\}.$$

It can be verified that the collection $\mathcal{M}$ does not have the W-property.

$\mathbf{n}^T = (-1\ 1)$ is a normal to $\mathscr{H}$. The products $\mathbf{n}^T\mathcal{M}^1 = \{3, -1\}$ and $\mathbf{n}^T\mathcal{M}^2 = \{-1, -3, 1\}$ have both positive and negative elements. Therefore, the cones formed by the vectors in $\mathcal{M}^1$ and $\mathcal{M}^2$ do not lie on the same side of the line $\mathscr{H}$. Hence, by Theorem 4.4 the pair $(\mathcal{M}, \mathscr{H})$ is not W-matrisable.

Now consider the another collection $\mathcal{N}$ such that

$$\mathcal{N}^1 = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \quad \mathcal{N}^2 = \left\{ \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \quad \mathcal{N} = \left\{ \mathcal{N}^1, \mathcal{N}^2 \right\}.$$

It can be verified that this collection $\mathcal{N}$ does not have the W-property.

The product $\mathbf{n}^T\mathcal{N}^1 = \{3, 1\}$, has only positive numbers. Therefore, the cone formed by the vectors of the set $\mathcal{N}^1$ lie on one side of the line (hyperplane) $\mathscr{H}$. By Theorem 4.4 the pair $(\mathcal{N}, \mathscr{H})$ is W-matrisable. We now construct a new collection $\widetilde{\mathcal{N}} \in \mathcal{N} + \mathscr{H}$ following the steps of Theorem 4.4 as follows:

$$\text{Let } \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ then } \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 4\mathbf{f} = 3\mathbf{w}. \quad \text{Also let } \widetilde{\mathbf{b}^1} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + 4\mathbf{f} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\text{Further let } \widetilde{\mathcal{N}^1} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \quad \widetilde{\mathcal{N}^2} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

Clearly, the collection $\widetilde{\mathcal{N}} = \left\{ \widetilde{\mathcal{N}^1}, \widetilde{\mathcal{N}^2} \right\} \in \mathcal{M} + \mathscr{H}$ has the W-property.

5. P-matrisability and W-matrisability by subspaces

In this section some conditions for solving the P-matrisability problem with under-actuation degree greater than one are obtained. This is generalised to W-matrisability problem. The following theorem gives a condition for P-matrisability with subspaces:

Theorem 5.1. *Let $D \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{m \times p}$ and $\text{rank}(F) = p$. Let the span of the columns of F be denoted by $\mathscr{H}^p$. Let $\alpha \subset \overline{m}$, such that for each $j \in \alpha$ one has $a_j \mathbf{e}^j \in D[\bullet, j] + \mathscr{H}^p$ and $a_j > 0$. Then the pair (D, F) is P-matrisable if and only if the pair $(D[\beta, \beta], F[\beta, \bullet])$ is P-matrisable, where $\beta = \overline{m} \setminus \alpha$.*

Proof. Let the hypothesis of the above theorem be satisfied. Therefore, there exists $\mathbf{h}^j \in \mathscr{H}^p$ such that $D[\bullet, j] + \mathbf{h}^j = a_j \mathbf{e}^j$ with $a_j > 0$ for all $j \in \alpha$. Let $G[\bullet, \alpha]$ be such that, $\mathbf{h}^j = FG[\bullet, j]$. If $M = D + FG$, then M is matrix whose columns are positive multiples of $\mathbf{e}^j$ for all $j \in \alpha$. So, by Lemma 2.1 the pair (D, F) is P-matrisable if and only if the pair $(D[\beta, \beta], F[\beta, \bullet])$ is P-matrisable. $\square$

The following condition for P-matrisability follows as a corollary of the above result.

Corollary 5.1. *Let $D \in \mathbb{R}^{m \times m}$ and $F \in \mathbb{R}^{m \times p}$. Assume without loss of generality that $\text{rank}(F) = p$. Let the span of the columns of F be denoted by $\mathscr{H}^p$. If there exists $\alpha \subset \overline{m}$, with $|\alpha| = m - p$ such that, for all $j \in \alpha$ one has, $a_j \mathbf{e}^j \in D[\bullet, j] + \mathscr{H}^p$, $a_j > 0$ and $\mathbf{e}^j \notin \mathscr{H}^p$ then the pair (D, F) is P-matrisable.*

Proof. By Lemma 3.2 , $F[\beta, \bullet]$ is invertible with $\beta = \overline{m} \setminus \alpha$ and $|\beta| = p$. The result now follows from Theorem 5.1 . $\square$

Let $D \in \mathbb{R}^{m \times m}$ and $F \in \mathbb{R}^{m \times p}$. Assume without loss of generality that $\text{rank}(F) = p$. Let $\mathscr{H}^p$ be the subspace spanned by the columns of F. Then it can be observed that, if there exists an $m - 1$ dimensional hyperplane $\mathscr{H}$ containing $\mathscr{H}^p$ and D is not P-matrisable with respect to $\mathscr{H}$, then D is also not P-matrisable with $\mathscr{H}^p$.

Let $\mathbf{n}^i \in \mathbb{R}^m$ for $i = 1, \ldots, m-p$ be a basis of the subspace normal to the p-dimensional subspace $\mathscr{H}^p$ spanned by the columns of F. Let the matrix N be such that $N[i, \bullet] = (\mathbf{n}^i)^T$ for $i = 1, \ldots, m - p$. Therefore, linear combinations of the rows of N represents all the normal vectors to the hyperplanes that contain the given p dimensional subspace $\mathscr{H}^p$. The formal version of the above observation is summarised in the form of a theorem below.

Theorem 5.2. *Given the pair (D, F), where $D \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{m \times p}$, let the matrix $N \in \mathbb{R}^{(m-p) \times m}$ be defined as above. Let $M = ND$.*

If there exists a vector $\mathbf{v} \in \mathbb{R}^{m-p}$ such that the sign patterns of the row matrices $\mathbf{v}^T N$ and $\mathbf{v}^T M$ disagree[3] *in all positions then the pair (D, F) is not P-matrisable.*

Proof. Follows from Theorem 3.1 and previous observation. $\square$

[3] Note that we assume that the signs of two variables disagree if one of them is zero.

We now derive a test for the above result: Let $\mathscr{C}_j = \text{PosCone}(M[\bullet, j], N[\bullet, j])$ for $j \in \overline{m}$. It is clear that $\mathscr{C}_j \subset \mathbb{R}^{m-p}$. Let $\mathscr{C}_j^* = \{\mathbf{v} \in \mathbb{R}^{m-p} \mid \langle \mathbf{v}, \mathbf{w} \rangle \geq 0 \, \forall \, \mathbf{w} \in \mathscr{C}_j\}$ denote the dual cone of $\mathscr{C}_j$. This is a solid cone but it may not be a proper cone in $\mathbb{R}^{m-p}$ for $m \geq 3$ and $0 < p \leq m-2$. The dual cone $\mathscr{C}_j^*$ contains the vectors $\mathbf{v} \in \mathbb{R}^{m-p}$ such that $\text{sgn}(\mathbf{v}^T M[\bullet, j])$ and $\text{sgn}(\mathbf{v}^T N[\bullet, j])$ are both positive. Similarly, the polar cone $-\mathscr{C}_j^*$ of $\mathscr{C}_j$, contains all the vectors $\mathbf{v} \in \mathbb{R}^{m-p}$, such that, $\text{sgn}(\mathbf{v}^T M[\bullet, j])$ and $\text{sgn}(\mathbf{v}^T N[\bullet, j])$ are both negative. Let $\mathscr{K}_j^* = \mathscr{C}_j^* \cup -\mathscr{C}_j^*$. Therefore, the set $\mathbb{R}^{m-p} \setminus \mathscr{K}_j^*$ denotes the range of $\mathbf{v}$'s where $\text{sgn}(\mathbf{v}^T M[\bullet, j]) \neq \text{sgn}(\mathbf{v}^T N[\bullet, j])$. Hence, by Theorem 3.1 , if $\cup_{j=1}^m \mathscr{K}_j^* \subsetneq \mathbb{R}^{m-p}$ then the pair (D, F) is not P-matrisable. Thus the following result is a test for the condition of P-matrisability by subspaces stated in Theorem 5.2 .

Theorem 5.3. *Let $D \in \mathbb{R}^{m \times m}$ and $F \in \mathbb{R}^{m \times p}$. Let $\mathscr{K}_j^*$ for $j \in \overline{m}$ be as defined above. If $\cup_{j=1}^m \mathscr{K}_j^* \neq \mathbb{R}^{m-p}$ then the pair (D, F) is not P-matrisable.*

Proof. Follows from above discussion and Theorem 5.2 . □

Following the argument for the pair (D, F), it is clear that, if there exists an $m-1$ dimensional hyperplane $\mathscr{H}$ containing $\mathscr{H}^p$ and the pair $(\mathcal{M}, \mathscr{H})$ is not W-matrisable, then the pair $(\mathcal{M}, \mathscr{H}^p)$ is also not W-matrisable.

Let the rows of N denote the basis for the normal space of $\mathscr{H}^p$. Let the notation $M_j = N\mathcal{M}^j$ be defined such that $M_j[\bullet, i] = Nm^{j,i}$ for $i \in \overline{m_j}$, where $m^{j,i} \in \mathcal{M}^j$, $m_j = |\mathcal{M}^j|$.

Let $\mathscr{C}_j = \text{PosCone}(M_j, N[\bullet, j])$ for $j \in \overline{m}$ be the positive cone generated by the columns of M_j and $N[\bullet, j]$. It is clear that $\mathscr{C}_j \subset \mathbb{R}^{m-p}$. Let $\mathscr{C}_j^* = \{\mathbf{v} \in \mathbb{R}^{m-p} \mid \langle \mathbf{v}, \mathbf{w} \rangle \geq 0 \, \forall \, \mathbf{w} \in \mathscr{C}_j\}$ denote the dual cone of $\mathscr{C}_j$. This is a solid cone but it may not be a proper cone in $\mathbb{R}^{m-p}$ for $m \geq 3$ and $0 < p \leq m-2$. The dual cone $\mathscr{C}_j^*$ contains the vectors $\mathbf{v} \in \mathbb{R}^{m-p}$ such that $\text{sgn}(\mathbf{v}^T M_j[\bullet, i])$ for all $i \in \overline{m_j}$ and $\text{sgn}(\mathbf{v}^T N[\bullet, j])$ are all positive. Similarly, $-\mathscr{C}_j^*$ contains the vectors $\mathbf{v} \in \mathbb{R}^{m-p}$ such that $\text{sgn}(\mathbf{v}^T M_j[\bullet, i])$ for all $i \in \overline{m_j}$ and $\text{sgn}(\mathbf{v}^T N[\bullet, j])$ are all negative. Let $\mathscr{K}_j^* = \mathscr{C}_j^* \cup -\mathscr{C}_j^*$. Therefore, the set $\mathbb{R}^{m-p} \setminus \mathscr{K}_j^*$ denotes the range of $\mathbf{v}$'s where the signs $\text{sgn}(\mathbf{v}^T M_j[\bullet, i])$ for all $i \in \overline{m_j}$ and $\text{sgn}(\mathbf{v}^T N[\bullet, j])$ do not match. Hence, by Theorem 4.4 , if $\cup_{j=1}^m \mathscr{K}_j^* \subsetneq \mathbb{R}^{m-p}$ then the pair $(\mathcal{M}, \mathscr{H})$ is not W-matrisable. Thus the following result is a test for the condition for W-matrisability.

Theorem 5.4. *Let the pair $(\mathcal{M}, \mathscr{H}^p)$, where $\mathscr{H}^p$ is a p-dimensional subspace be given. Let $\mathscr{K}_j^*$ be as defined above. If $\cup_{j=1}^m \mathscr{K}_j^* \neq \mathbb{R}^{m-p}$ then the pair $(\mathcal{M}, \mathscr{H}^p)$ is not W-matrisable.*

Proof. Follows from above discussion. □

As checking P-matrisability in the general case is a hard problem, in the next section a subclass of P-matrices is considered. It turns out that one can obtain polynomial time algorithm to check if the given matrices can be converted to a P-matrix in this subclass.

6. P-matrisability for special classes

In [3] it has been shown using graph theoretic interpretation that a subclass of matrices can be shown to be from P-matrix class by only looking at the signs of their components. For example all matrices with sign pattern

$$\begin{bmatrix} + & + \\ - & + \end{bmatrix}, \begin{bmatrix} + & - \\ + & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & + & + \\ 0 & - & + \end{bmatrix}$$

can be shown to be P-matrices. Such matrices are called signed P-matrices. Clearly, the decision algorithm for P-matrices from such classes has polynomial time complexity.

In the general P-matrisability problem, the existence of matrix G such that $D + FG$ is a P-matrix was investigated. It is interesting to note that the complexity of any algorithm for P-matrisability is expected to be at least (complement) co-NP-complete. This is because, determining whether a matrix is a P-matrix is also co-NP-complete. That is, the algorithms for determining whether a matrix is a P-matrix or, whether it is not a P-matrix are both NP complete. Therefore, one would like to know if there exists a subclass of P-matrices for which one has a faster (polynomial time) algorithm for checking P-matrisability. Fortunately, one such class of P-matrices is the signed P-matrix class.

The P-matrisability problem for this subclass is formulated using linear inequalities as follows:

Problem 6.1. Let $D \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times p}$. Let $S \in \mathbb{R}^{m\times m}$ be a given P-matrix sign pattern matrix. Compute a matrix $G \in \mathbb{R}^{p\times m}$ satisfying the following inequalities:

$$D[i,j] + F[i,\bullet]G[\bullet,j] \begin{cases} > 0 \text{ if } \operatorname{sgn}(S[i,j]) = +, \\ < 0 \text{ if } \operatorname{sgn}(S[i,j]) = -, \\ = 0 \text{ if } \operatorname{sgn}(S[i,j]) = 0. \end{cases} \tag{7}$$

It is clear that these inequalities involve m^2 inequality relations in pm unknown components of G. It is well known that the solution of linear inequalities can be obtained in polynomial time in the number of relations, see for example [20]. Thus for each signed matrix S the above P-matrisability problem can be decided in polynomial time.

7. Separation theorem for P-matrix and W-property

In this section an alternate proof that P-matrices satisfy the separation property [4] is provided. Then a generalisation of the notion separation property to a collection satisfying the W-property is obtained.

The notion of complementary set of column vectors with respect to a square matrix is defined as follows: Given a matrix $M \in \mathbb{R}^{m\times m}$ the set $\{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^m\}$ is called a complementary set of column vectors if $\mathbf{v}^i \in \{M[\bullet,i], I[\bullet,i]\}$ where I is the identity matrix of order m. Note that a complementary set contains columns of some column representative matrix of the pair (M, I). A sub-complementary set is obtained by leaving out one column vector from the complementary set. Leaving out $\mathbf{v}^j$ results in not including the pair $\{M[\bullet,j], I[\bullet,j]\}$ which is called the left out complementary pair of the sub complementary set $\{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$.

The strict separation property of P-matrices is stated below:

Theorem 7.1. *Let $M \in \mathbb{R}^{m\times m}$. Then M is a P-matrix if and only if for every sub complementary set $\mathcal{K} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$ of M, the left out complementary pair $\{M[\bullet,j], I[\bullet,j]\}$ lie on the same open half space formed by the hyperplane* span $\{\mathcal{K}\}$.

Proof. (**If**): Let M be a P-matrix. Let $\mathcal{K} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$ be a sub complementary set of columns of M. Let $D \in \mathbb{R}^{m\times m}$ be such that $D[\bullet,j] = M[\bullet,j]$ and remaining columns set to $\mathbf{0}$. Define $F \in \mathbb{R}^{m\times(m-1)}$ to be $F[\bullet,i] = \mathbf{v}^i$ for $i \in \overline{m} \setminus \{j\}$. As M is a P-matrix, the matrix $[\mathbf{v}^1, \ldots, M[\bullet,j] \ldots, \mathbf{v}^m]$ is also a P-matrix. Therefore, pair (D, F) is P-matrisable. Thus by Theorem 3.1 , the (left out complementary) vector pair $\{M[\bullet,j], I[\bullet,j]\}$ lie on the same open half space defined by the vectors in $\mathcal{K}$. This is true for any $j \in \overline{m}$.

(**Only if**): See [4]. $\square$

Paralleling the definition of complementary set of columns for a matrix we define it for a collection $\mathcal{M}$ as follows: A set of vectors $\{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^m\}$ is called a complementary set of columns of a collection $\mathcal{M}$, if $\mathbf{v}^i \in \mathcal{M}^i$ for $i \in \overline{m}$. It is clear that a complementary set of a collection $\mathcal{M}$ contains the columns of some representative matrix of $\mathcal{M}$. By leaving out a vector $\mathbf{v}^j$ for some $j \in \overline{m}$ from a complementary set, one obtains the sub complementary set. The set $\mathcal{M}^j$ from which $\mathbf{v}^j$ was not included in a complementary set is called the left out complementary set of the sub complementary set $\{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$.

The strict separation property for the W-property is stated below:

Theorem 7.2. *The following are equivalent:*

1. *A collection $\mathcal{M}$ has the W-property.*
2. *For each $j \in \overline{m}$, the set $\mathcal{M}^j$ lies in an open half space defined by the hyperplanes $\mathcal{K}$. These $\mathcal{K}$s are hyperplanes spanned by sub complementary sets $\{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$, where $\mathbf{v}^l \in \mathcal{M}^l$ and $l \in \overline{m} \setminus \{j\}$.*

Proof. (**If**): Let a collection $\mathcal{M}$ have the W-property. Let

$$\mathcal{K} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$$

be a sub complementary set of the collection $\mathcal{M}$ for some $j \in \overline{m}$. Let the hyperplane $\mathscr{L}$ be spanned by the vectors in $\mathcal{K}$. Since the collection $\mathcal{M}$ has the W-property, the subspace $\mathscr{L}$ is $m-1$ dimensional and the pair $(\mathcal{M}, \mathscr{L})$ is W-matrisable. By Theorem 4.4 and Lemma 4.7 therefore, the left out complementary set of vectors $\mathcal{M}^j$ all lie on the same open half space defined by $\mathscr{L}$.
(**Only if**): Let $\mathcal{M}$ be a collection satisfying the strict separation property. That is, for every $j \in \overline{m}$ the vectors in the left out complementary set $\mathcal{M}^j$ all lie in the same open half space of the hyperplane defined by the sub-complementary set $\mathcal{K} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, \ldots, \mathbf{v}^m\}$. This implies that all the representative matrices of the collection $\mathcal{M}$ has a non-zero determinant. It is now enough to show that, any two arbitrary representative matrices R and S of the collection $\mathcal{M}$ have the same determinant sign. Let $\mathbf{r}^{\mathbf{j}}$ and $\mathbf{s}^{\mathbf{j}}$ represent the jth column of R and S respectively for $j \in \overline{m}$. Let the notation $\mathcal{R}^j$ represent the sub-complementary set obtained by dropping the jth column of any column representative matrix R. Consider the sub-complementary set $\mathcal{R}^1 = \mathcal{R}_0^1$ of the representative matrix R. Since the m-tuple $\mathcal{M}$ satisfies the strict separation property, the column vectors $\mathbf{r}^{\mathbf{1}}, \mathbf{s}^{\mathbf{1}}$ of the matrices R and S respectively lie on the same side of the hyperplane formed by the vectors in $\mathcal{R}_0^1$. Therefore, the sign of the determinant of the matrix $R = R_0$ and the matrix R_1 obtained after replacing the first column of R by the first column $\mathbf{s}^{\mathbf{1}}$ of S, will remain the same.

Now let R_k denote the matrix obtained after replacing the kth column of R_{k-1} by the vector $\mathbf{s}^{\mathbf{k}}$ (kth column of matrix S) for some $k > 1$ and $k \in \overline{m}$. It can be verified that the first k columns of R_k and S and the remaining columns of R_k and R are identical. Note that the matrices R_{k-1} and R_k are representative matrices of the collection $\mathcal{M}$. It is claimed that the signs of determinants of the matrices R_{k-1} and R_k are the same. For this consider the sub-complementary set $\mathcal{R}_{k-1}^k$. From the hypothesis it is clear that the kth column vector $\mathbf{r}^{\mathbf{k}}$ of the matrix R_{k-1} (and hence R) and $\mathbf{s}^{\mathbf{k}}$ of the matrix S both lie on the same open half space defined by the hyperplane spanned by the sub-complementary set $\mathcal{R}_{k-1}^k$. Therefore, the signs of the determinants of the matrices R_{k-1} and R_k are the same. Proceeding in this manner one obtains the representative matrix sequence $\{R = R_0, R_1, \ldots, R_m = S\}$ all having the same determinant sign. This means signs of determinants of the representative matrices R and S are the same. Since this is true for arbitrary representative matrices R and S the result follows. □

Thus a geometric characterisation of the W-property which is a generalisation of the separation theorem for P-matrices has been obtained.

8. Conclusions

In this paper the problem of generating P-matrices from a given matrix pair and the problem of generating matrix k-tuples satisfying column-W property from a matrix $(k+1)$-tuple was considered. The motivation for considering these problems arose from the feedback regularisation of LCS and a class of PLS. The notion of P-matrisability and W-matrisability was introduced to solve these problems. A necessary and sufficient condition for P-matrisability by hyperplanes was obtained. Then the W-matrisability of matrix k-tuples was introduced and linked with the W-property of a collection. A necessary and sufficient condition for W-matrisability of a collection by hyperplanes was obtained. A necessary condition and an independent sufficient condition for P-matrisability and W-matrisability with lower dimensional subspace were also obtained. Then a subclass of the P-matrisability problem which could be solved in polynomial time was given. In the final section an alternative proof, that the P-matrix satisfies the separation property was put forward. Finally, the concept of separation principle for P-matrices was generalised to a collection satisfying the W-property.

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